Poisson quasi-Nijenhuis structures with background

PAULO ANTUNES
CMUC, DEPARTMENT OF MATHEMATICS
UNIVERSITY OF COIMBRA
3001-454 COIMBRA
PORTUGAL
pantunes@mat.uc.pt

Abstract

We define the Poisson quasi-Nijenhuis structures with background on Lie algebroids and we prove that to any generalized complex structure on a Courant algebroid which is the double of a Lie algebroid is associated such a structure. We prove that any Lie algebroid with a Poisson quasi-Nijenhuis structure with background constitutes, with its dual, a quasi-Lie bialgebroid. We also prove that any pair (π,ω) of a Poisson bivector and a 2-form induces a Poisson quasi-Nijenhuis structure with background and we observe that particular cases correspond to already known compatibilities between π and ω.

Introduction

The aim of this work[1] is to define the notion of Poisson quasi-Nijenhuis structure on a Lie algebroid with a (closed) 3-form background. The Poisson quasi-Nijenhuis structures (without background) were introduced by Stiénon and Xu in [14] on the tangent Lie algebroid and then on any Lie algebroid by Caseiro et al. in [1]. The case with background was already studied by Zucchini [19] but we remarked that a condition is missing in the definition proposed there. This extra condition was already in [14] and appears naturally in this work when we ask for some structures to be integrable (or some brackets to verify the Jacobi identity).

In this work we will use a supermanifold approach [16, 12] to describe Lie algebroid structures. Let us consider a vector bundle $A \to M$ and change the parity of the fiber coordinates (considering them odd), then we obtain a supermanifold denoted by ΠA. The algebra of functions on ΠA, which are polynomial in the fibre coordinates, is denoted by $C^\infty(\Pi A)$ and coincides with $\Omega(A) := \Gamma(\bigwedge^\bullet A^*)$, the exterior algebra of A-forms. Let consider a Lie algebroid structure on A given by d, a degree 1 derivation of $\Omega(A)$ such that $d^2 = 0$. In this supermanifold setting, d is a vector field on ΠA and can be seen as the derivation defined by a hamiltonian on ΠA, i.e. an element $\mu \in C^\infty(T^*\Pi A)$. Then $d = \{\mu, .\})$, where the so-called big bracket [5], $\{.,.\}$, is the canonical Poisson bracket on the symplectic supermanifold $T^*\Pi A$. The condition $d^2 = 0$ is equivalent to $\{\mu,\mu\} = 0$.

To each $f \in C^\infty(T^*\Pi A)$ is associated a bidegree (ε,δ). In fact, since using Legendre transform (see [10]) $T^*(\Pi A) \cong T^*(\Pi A^*)$, we can define ε (resp. δ) as the polynomial degree of

[1] This paper was presented as a poster at the "Poisson 2008" conference at the EPFL in Lausanne in July 2008.

f in the fibre coordinates of the vector bundle $T^*(\Pi A) \to \Pi A$ (resp. $T^*(\Pi A) \to \Pi A^*$). We define the shifted bidegree of f as the pair $(\varepsilon-1, \delta-1)$ and the total shifted bidegree as the sum $(\varepsilon-1)+(\delta-1) = \varepsilon+\delta-2$. Then a Lie algebroid structure in A is a hamiltonian $\mu \in C^\infty(T^*\Pi A)$ of shifted bidegree $(0,1)$ such that $\{\mu,\mu\} = 0$.

Instead of the expression "with background" used here, some authors use "twisted", or in Physics, "H-flux". In this work, our choice was motivated by the result of the proposition 4.2. In fact, we prove there that a particular class of Poisson quasi-Nijenhuis structure with background is obtained by twisting, in a way explained in [11, 15, 7], a Lie algebroid structure by a Poisson bivector and then by a 2-form. Therefore, to avoid confusion, we will use the word "twist" only when we are dealing with twisting by a 2-form or a bivector as in [11, 15, 7].

The content of this paper is as follows. In the first section we recall some basic definitions about Nijenhuis tensors, Poisson bivectors and Poisson Nijenhuis structures on a Lie algebroid and give the corresponding expression in the supermanifold approach. Then, in the second section, we introduce the notion of Poisson quasi-Nijenhuis structure with a 3-form background H, on a Lie algebroid (A,μ). We prove that any complex structure (or more generally any c.p.s. structure, see definition 2.2) on $(A\oplus A^*, \mu+H)$ induces such a structure. In the third section we generalize a result from [14, 1] and prove that any Poisson quasi-Nijenhuis structure with background on A induces a Lie quasi-bialgebroid on (A^*,A). Finally, in the last section we study Poisson quasi-Nijenhuis structures with background defined by a pair (π,ω) of a Poisson bivector and a 2-form. We observe that already known compatible pairs such that complementary 2-forms for Poisson bivectors [17], Hitchin pairs [2] and $P\Omega$-structures or ΩN-structures [9] are all particular examples of Poisson quasi-Nijenhuis structures with background.

1 Basic definitions

In this section we will recall known structures such as Poisson Nijenhuis structures on a Lie algebroid A and give their expression in terms of big bracket and polynomial functions on $T^*\Pi A$.

Let $(A,[.,.],\rho)$ be a Lie algebroid over a smooth manifold M. The Lie algebroid structure, $([.,.],\rho)$, can be seen as a function $\mu \in C^\infty(T^*\Pi A)$, of shifted bidegree $(0,1)$, such that $\{\mu,\mu\} = 0$.

Consider a $(1,1)$-tensor $N \in \Gamma(A\otimes A^*)$. The Nijenhuis torsion of N is defined by

$$T_N(X,Y) = [NX,NY] - N\left([NX,Y]+[X,NY]-N[X,Y]\right).$$

In terms of big bracket and elements of $C^\infty(T^*\Pi A)$, the Nijenhuis torsion is given by

$$T_N = \frac{1}{2}\left(\{N,\{N,\mu\}\} - \left\{N^2,\mu\right\}\right). \tag{1}$$

If $T_N = 0$, N is said to be a Nijenhuis tensor and in this case we define a new Lie algebroid structure on A given by

$$\begin{cases} [X,Y]_N = [NX,Y]+[X,NY]-N[X,Y], \qquad X,Y\in\Gamma(A), \\ \rho_N = \rho\circ N. \end{cases} \tag{2}$$

In the supermanifold setting, the structure $([.,.]_N, \rho_N)$ is given by $\{N,\mu\} \in C^\infty(T^*\Pi A)$. We denote by d_N the degree 1 derivation of $\Omega(A)$ induced by this Lie algebroid structure. Then

$$d_N = \{\{N,\mu\},.\}.$$

A bivector $\pi \in \Gamma(\bigwedge^2 A)$ is said to be Poisson if $[\pi,\pi]_{SN} = 0$, where $[.,.]_{SN}$ is the Schouten-Nijenhuis bracket naturally defined on $\Gamma(\bigwedge^\bullet A)$. If π is a Poisson bivector we define a Lie algebroid structure on A^* by setting

$$\begin{cases} [\alpha,\beta]_\pi = \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - d(\pi(\alpha,\beta)), & \alpha,\beta \in \Gamma(A^*), \\ \rho_\pi = \rho\circ\pi^\sharp. \end{cases} \tag{3}$$

In the supermanifold setting, the structure $([.,.]_\pi, \rho_\pi)$ is given by $\{\pi,\mu\} \in C^\infty(T^*\Pi A)$.

Definition 1.1. A Poisson bivector π and a Nijenhuis tensor N are said to be *compatibles* if

$$\begin{cases} N\circ\pi^\sharp = \pi^\sharp \circ {}^tN, \\ C_{\pi,N} = 0, \end{cases}$$

with

$$C_{\pi,N} = ([.,.]_N)_\pi - ([.,.]_\pi)_{^tN}\,,$$

a $C^\infty(M)$-bilinear bracket on $\Gamma(A^*)$. When π and N are compatible, (A,π,N) constitutes a *Poisson Nijenhuis Lie algebroid*.

In the supermanifold setting, we have

$$C_{\pi,N} = \{\pi,\{N,\mu\}\} + \{N,\{\pi,\mu\}\}.$$

Theorem 1.2. *If (A,π,N) is a Poisson Nijenhuis Lie algebroid, then (A_N, A^*_π) is a Lie bialgebroid, where A_N and A^*_π are the Lie algebroids defined respectively by (2) and (3).*

Remark 1.3. When $A = TM$ and μ is the standard Lie algebroid structure, the implication of the previous theorem becomes an equivalence (see [6]).

The Lie bialgebroid (A_N, A^*_π) induces a Courant algebroid structure in $A\oplus A^*$ [8, 10] which is given in the supermanifold setting by

$$S = \{\pi,\mu\} + \{N,\mu\} = \{\pi+N,\mu\}\,.$$

2 Poisson quasi-Nijenhuis with background and generalized geometry

Let S be a Courant algebroid structure on $A\oplus A^*$, i.e., $S \in C^\infty(T^*\Pi A)$ is of total shifted degree 1 and $\{S,S\} = 0$. Consider also a $(1,1)$-tensor J on $A\oplus A^*$, seen as a map $J : A\oplus A^* \to A\oplus A^*$. We call J *orthogonal* if

$$< J(\mathcal{X}),\mathcal{Y} > + < \mathcal{X}, J(\mathcal{Y}) >= 0,$$

for all $\mathcal{X},\mathcal{Y} \in \Gamma(A\oplus A^*)$, with $< .,. >$ defined by $< X+\alpha, Y+\beta >= \beta(X)+\alpha(Y)$ for all $X,Y \in \Gamma A$, $\alpha,\beta \in \Gamma A^*$.

As in the Lie algebroid case, we can define a new bracket $[.,.]_J$ deforming by J the Courant structure on $A\oplus A^*$ by setting

$$[\mathcal{X},\mathcal{Y}]_J = [J\mathcal{X},\mathcal{Y}] + [\mathcal{X},J\mathcal{Y}] - J[\mathcal{X},\mathcal{Y}],$$

for all $\mathcal{X},\mathcal{Y} \in \Gamma(A\oplus A^*)$, where $[.,.]$ is the Dorfman bracket on $A\oplus A^*$. When J is an orthogonal $(1,1)$-tensor on $A\oplus A^*$, this deformed bracket is given by the hamiltonian

$$S_J := \{J,\mu\} \in C^\infty(T^*\Pi A).$$

We define also the Nijenhuis torsion of J,

$$T_J(\mathcal{X},\mathcal{Y}) = [J\mathcal{X}, J\mathcal{Y}] - J\left([\mathcal{X},\mathcal{Y}]_J\right),$$

for all $\mathcal{X},\mathcal{Y} \in \Gamma(A\oplus A^*)$.

Proposition 2.1. *1. The hamiltonian S_J defines a Courant structure on $A\oplus A^*$ if and only if $\{S,T_J\} = 0$.*

2. J is a Courant morphism from $(A\oplus A^*, S_J)$ to $(A\oplus A^*, S)$ if and only if $T_J = 0$.

Definition 2.2. An orthogonal $(1,1)$-tensor J, on $A\oplus A^*$, is an *almost c.p.s. structure* if $J^2 = \lambda\, id_{A\oplus A^*}$, with $\lambda \in \{-1,0,1\}$. The almost c.p.s. structure J is integrable when $T_J = 0$.

The abbreviation "c.p.s." is due to Vaisman [18] and corresponds to the three different structures we are considering: if $\lambda = -1$, J is an almost complex structure; if $\lambda = 1$, J is an almost product structure; and if $\lambda = 0$, J is an almost subtangent structure.

As was noticed in [2, 18], J is an almost c.p.s. structure if and only if J can be represented in a matrix form by

$$J\begin{pmatrix} X \\ \alpha \end{pmatrix} = \begin{pmatrix} N & \pi^\sharp \\ \sigma^\flat & -{}^tN \end{pmatrix}\begin{pmatrix} X \\ \alpha \end{pmatrix} \tag{4}$$

for all $X \in \Gamma(A)$ and $\alpha \in \Gamma(A^*)$, where $\pi \in \Gamma(\bigwedge^2 A), \sigma \in \Gamma(\bigwedge^2 A^*)$ and $N \in \Gamma(A\otimes A^*)$ satisfy

$$\begin{cases} N\circ\pi^\sharp = \pi^\sharp\circ {}^tN, \\ \sigma^\flat\circ N = {}^tN\circ\sigma^\flat, \\ N^2 + \pi^\sharp\circ\sigma^\flat = \lambda\, id_A. \end{cases}$$

In the supermanifold setting,

$$J = \pi + N + \sigma$$

in the sense that $J(.) = \{., \pi + N + \sigma\}$. Moreover, the integrability condition of an almost c.p.s. structure, $T_J = 0$, is expressed by [3]

$$\{\{J,S\},J\} + \lambda S = 0. \tag{5}$$

Let us now consider the case where $S = \mu + H$, where $\mu \in C^\infty(T^*\Pi A)$ defines a Lie algebroid structure on A, and $H \in \Gamma(\bigwedge^3 A^*)$ is a closed 3-form. Then $\{S,S\} = 0$ and S defines a Courant algebroid structure on $A\oplus A^*$.

The goal of this section is to relate c.p.s. structures on $(A\oplus A^*, \mu + H)$ with the Poisson quasi-Nijenhuis structures with background which we now define.

Definition 2.3. A *Poisson quasi-Nijenhuis structure with background* on a Lie algebroid A is a quadruple (π, N, ψ, H) where $\pi \in \Gamma(\bigwedge^2 A)$, $N \in \Gamma(A\otimes A^*)$, $\psi \in \Gamma(\bigwedge^3 A^*)$ and $H \in \Gamma(\bigwedge^3 A^*)$ are such that $N\circ\pi^\sharp = \pi^\sharp\circ {}^tN$, $d\psi = 0$, $dH = 0$ and verify the conditions

$$\begin{cases} \pi \text{ is a Poisson bivector}, \\ C_{\pi,N}(\alpha,\beta) = 2\, i_{\pi^\sharp\alpha\wedge\pi^\sharp\beta} H, \\ T_N(X,Y) = \pi^\sharp(i_{NX\wedge Y}H - i_{NY\wedge X}H + i_{X\wedge Y}\psi), \\ \mathrm{d}_N\psi = \mathrm{d}\mathcal{H}, \end{cases} \tag{6}$$

for all $X,Y \in \Gamma(A)$, $\alpha,\beta \in \Gamma(A^*)$ and where $\mathcal{H}$ is the 3-form defined by

$$\mathcal{H}(X,Y,Z) = \circlearrowleft_{X,Y,Z} H(NX,NY,Z), \tag{7}$$

for all $X,Y,Z \in \Gamma(A)$.

Observation 2.4. 1. In terms of big bracket and elements of $C^\infty(T^*\Pi A)$, the conditions (6) correspond to

$$\begin{cases} \{\{\pi,\mu\},\pi\} = 0, \\ \{\{\pi,\mu\},N\} + \{\{N,\mu\},\pi\} + \{\{\pi,H\},\pi\} = 0, \\ \{\{N,\mu\},N\} + \{N^2,\mu\} - 2\{\pi,\psi\} + \{\{\pi,H\},N\} + \{\{N,H\},\pi\} = 0, \\ 2\{\{N,\mu\},\psi\} = \{\mu,\{N,\{N,H\}\} - \{N^2,H\}\}. \end{cases} \tag{8}$$

2. If $H = 0$ we recover the Poisson quasi-Nijenhuis structures defined in [1, 14].

3. The last condition of (6) is missing in the definition proposed by Zucchini [19]. In our study this condition appears naturally and is necessary in order to include the case without background, described in [1, 14].

Theorem 2.5. *If an endomorphism J, defined by (4), is a c.p.s. structure on* $(A \oplus A^*, \mu + H)$ *then* $(\pi, N, -d\sigma, H)$ *is a Poisson quasi-Nijenhuis structure with background on A.*

Proof. The result follows directly by writing the integrability condition (5) with $J = \pi + N + \sigma$ and $S = \mu + H$. Using the bilinearity of $\{.,.\}$ and taking into account the bidegree of each term we obtain the following system of equations

$$\begin{cases} \{\{\pi,\mu\},\pi\} = 0, \\ \{\{\pi,\mu\},N\} + \{\{N,\mu\},\pi\} + \{\{\pi,H\},\pi\} = 0, \\ \{\{N,\mu\},N\} + 2\{\pi,\{\mu,\sigma\}\} + \{\{\pi,\sigma\},\mu\} + \{\{\pi,H\},N\} + \{\{N,H\},\pi\} + \lambda\mu = 0, \\ \{\{N,\mu\},\sigma\} + \{\{\sigma,\mu\},N\} + \{\{N,H\},N\} + \{\{\pi,\sigma\},H\} + \lambda H = 0. \end{cases}$$

In the last two equations of the system we now use the algebraic conditions for J to be a c.p.s. structure and more precisely the condition $N^2 + \pi^\sharp \circ \sigma^\flat = \lambda\, id_A$ which is written in terms of big bracket and elements of $C^\infty(T^*\Pi A)$ as

$$\{\pi,\sigma\} = N^2 - \lambda\, id_A.$$

We obtain

$$\begin{cases} \{\{\pi,\mu\},\pi\} = 0, \\ \{\{\pi,\mu\},N\} + \{\{N,\mu\},\pi\} + \{\{\pi,H\},\pi\} = 0, \\ \{\{N,\mu\},N\} + \{N^2,\mu\} + 2\{\pi,\{\mu,\sigma\}\} + \{\{\pi,H\},N\} + \{\{N,H\},\pi\} = 0, \\ \{\{N,\mu\},\sigma\} + \{\{\sigma,\mu\},N\} + \{\{N,H\},N\} + \{N^2,H\} - 2\lambda H = 0. \end{cases}$$

The proof is achieved after interpreting the previous system of equations as

$$\begin{cases} \pi \text{ is a Poisson bivector}, \\ C_{\pi,N}(\alpha,\beta) = 2\, i_{\pi^\sharp\alpha\wedge\pi^\sharp\beta} H, \\ T_N(X,Y) = \pi^\sharp(i_{NX\wedge Y}H - i_{NY\wedge X}H - i_{X\wedge Y}d\sigma), \\ 2 i_N d\sigma - d(i_N\sigma) = 2(\mathcal{H} + \lambda H), \end{cases} \tag{9}$$

for all $X, Y \in \Gamma(A)$ and $\alpha, \beta \in \Gamma(A^*)$ and $\mathcal{H}$ defined by Equation (7). $\square$

Remark 2.6. In [18], Vaisman studied the integrability of almost c.p.s. structures on $TM \oplus T^*M$ considering both the usual Courant bracket, and also the case with a 3-form background. The conditions that he obtained in Remark 1.5 of [18] coincide with the system of conditions (9).

Note that, in the previous proof, the last equation of (9) is only a sufficient condition for the last equation of (6). We can get an equivalence if we impose additional conditions on the quadruple (π, N, σ, H).

Theorem 2.7. *An endomorphism J, defined by (4), is a c.p.s. structure on $(A \oplus A^*, \mu + H)$ if and only if $(\pi, N, -d\sigma, H)$ is a Poisson quasi-Nijenhuis structure on A with background H such that*

$$\begin{cases} N^2 + \pi^\sharp \circ \sigma^\flat = \lambda \, id_A, \\ \sigma^\flat \circ N = {}^tN \circ \sigma^\flat, \\ 2(i_N d\sigma - \mathcal{H}) = d(i_N \sigma) + 2\lambda H. \end{cases}$$

3 Poisson quasi-Nijenhuis with background and Lie quasi-bialgebroids

In this section we will generalize a result proved for structures without background in [14, 1]. Let (A, μ) be a Lie algebroid over a smooth manifold M.

Definition 3.1. A Lie quasi-bialgebroid is a triple (A, δ, φ) where A is a Lie algebroid, δ is a degree one derivation of the Gerstenhaber algebra $(\Gamma(\bigwedge^\bullet A), \wedge, [.,.])$ and $\varphi \in \Gamma(\bigwedge^3 A)$ is such that $\delta^2 = [\varphi, .]$ and $\delta\varphi = 0$.

The main result of the section is the following

Theorem 3.2. *If (π, N, ψ, H) is a Poisson quasi-Nijenhuis structure with background on A then $(A^*_\pi,\ d_N^H,\ \psi + i_N H)$ is a Lie quasi-bialgebroid, where $d_N^H(\alpha) = d_N(\alpha) - i_{\pi^\sharp(\alpha)} H$, for all $\alpha \in \Gamma(A^*)$.*

Proof. The hamiltonian on $C^\infty(T^*\Pi A)$ which induces the structure $(A^*_\pi,\ d_N^H,\ \psi + i_N H)$ is $\widetilde{S} = \{\pi + N, \mu + H\} + \psi$. Considering the bidegree of each term, the equation $\left\{\widetilde{S}, \widetilde{S}\right\} = 0$ is equivalent to

$$\begin{cases} \{\{\pi,\mu\},\{\pi,\mu\}\} = 0, \\ \{\{\pi,\mu\},\{\pi,H\}\} + \{\{\pi,\mu\},\{N,\mu\}\} = 0, \\ \{\{\pi,H\},\{\pi,H\}\} + \{\{N,\mu\},\{N,\mu\}\} + 2\{\{\pi,\mu\},\{N,H\}\} + \\ \qquad\qquad + 2\{\{\pi,H\},\{N,\mu\}\} + 2\{\{\pi,\mu\},\psi\} = 0, \\ \{\{\pi,H\},\{N,H\}\} + \{\{N,\mu\},\{N,H\}\} + \{\{\pi,H\},\psi\} + \{\{N,\mu\},\psi\} = 0. \end{cases} \tag{10}$$

It is now straightforward to observe that the system of equations (8) implies the system (10). ☐

Corollary 3.3. *If $(\pi, N, -d\sigma, H)$ is a Poisson quasi-Nijenhuis structure with background on A then $\{\pi + N + \sigma, \mu + H\}$ is a structure of Lie quasi-bialgebroid on (A^*, A) or equivalently of a Courant algebroid on $A \oplus A^*$.*

Proof. If we consider $\psi = -d\sigma$ in the previous proof, then $\widetilde{S} = \{\pi + N + \sigma, \mu + H\}$ and we prove that $\left\{\widetilde{S}, \widetilde{S}\right\} = 0$. ☐

Remark 3.4. In the corollary above, $J = \pi + N + \sigma$ is not necessarily integrable, i.e., the Nijenhuis torsion T_J may not vanish (see necessary conditions in theorem 2.7). But the previous corollary proves that the deformed structure S_J defines a Courant algebroid structure in $A \oplus A^*$, i.e., that $\{S, T_J\} = 0$ (see proposition 2.1).

4 Poisson quasi-Nijenhuis with background and compatible second order tensors

In this section we shall consider $\pi \in \Gamma(\bigwedge^2 A)$ a Poisson bivector and a 2-form $\omega \in \Gamma(\bigwedge^2 A^*)$. Let us denote

$$\begin{array}{ll} \pi^\sharp(\alpha) = \pi(\alpha,.), \quad \forall \alpha \in \Gamma(A^*), & \omega^\flat(X) = \omega(X,.), \quad \forall X \in \Gamma(A), \\ N = \pi^\sharp \circ \omega^\flat, & \omega_N = \omega(N.,.). \end{array}$$

Then the main result of this section is the following

Theorem 4.1. *The quadruple $(\pi, N, d\omega_N, -d\omega)$ is a Poisson quasi-Nijenhuis structure with background on A.*

Proof. Let us denote $\psi = d\omega_N$ and $H = -d\omega$. In terms of elements of $C^\infty(T^*\Pi A)$, we have the following correspondences

$$\begin{cases} N = \{\omega, \pi\}, \\ \psi = \frac{1}{2}\{\mu, \{N, \omega\}\}, \\ H = \{\omega, \mu\}. \end{cases}$$

We easily check that ψ and H are closed and that $N \circ \pi^\sharp = \pi^\sharp \circ {}^t N$. To prove that (π, N, ψ, H) is a Poisson quasi-Nijenhuis structure with background we need to verify the set of conditions (6) (or equivalently the conditions (8)).

1. π is a Poisson bivector by assumption.

2. Let us start from the fact that π is a Poisson bivector, i.e., that

$$\{\{\pi, \mu\}, \pi\} = 0$$

and apply $\{\omega, .\}$ to both sides. We get

$$\{\omega, \{\{\pi, \mu\}, \pi\}\} = 0.$$

We use the Jacobi identity and obtain

$$\{\{\omega, \{\pi, \mu\}\}, \pi\} + \{\{\pi, \mu\}, \{\omega, \pi\}\} = 0$$

and using once more the Jacobi identity in the first term of l.h.s. we have

$$\{\{\{\omega, \pi\}, \mu\}, \pi\} + \{\{\pi, \{\omega, \mu\}\}, \pi\} + \{\{\pi, \mu\}, \{\omega, \pi\}\} = 0,$$

which is the second condition of (8)

$$\{\{N, \mu\}, \pi\} + \{\{\pi, H\}, \pi\} + \{\{\pi, \mu\}, N\} = 0.$$

3. As above, we start from the previous condition

$$\{\{N, \mu\}, \pi\} + \{\{\pi, H\}, \pi\} + \{\{\pi, \mu\}, N\} = 0,$$

and apply $\{\omega, .\}$ to both sides. We obtain

$$\{\omega, \{\{N, \mu\}, \pi\}\} + \{\omega, \{\{\pi, H\}, \pi\}\} + \{\omega, \{\{\pi, \mu\}, N\}\} = 0,$$

and using the Jacobi identity twice we get the required equation

$$\{\{N, \mu\}, N\} + \left\{N^2, \mu\right\} - 2\{\pi, \psi\} + \{\{\pi, H\}, N\} + \{\{N, H\}, \pi\} = 0.$$

4. The way of proving this condition is the same as above. We start from the previous condition and apply $\{\omega, .\}$ to both sides. Then, using the Jacobi identity, we get

$$\{\{N,H\},N\}+\{N^2,H\}-2\{N,\psi\}-\{\{N^2,\omega\},\mu\}=0. \tag{11}$$

Finally, applying $\{\mu, .\}$ we obtain

$$\{\mu,\{\{N,H\},N\}+\{N^2,H\}\}-2\{\mu,\{N,\psi\}\}=0.$$

Using once more the Jacobi identity and the fact that ψ is closed we get

$$2\{\{N,\mu\},\psi\}=\{\mu,\{N,\{N,H\}\}-\{N^2,H\}\}.$$

□

The proof of the previous theorem suggests that starting from a Poisson bivector and composing iteratively, in a certain way, with a 2-form we get all the conditions of the definition of a Poisson quasi-Nijenhuis structure with background. The precise way to describe this fact is using the twist of a structure by a bivector or a 2-form as in [11, 15, 7].

Proposition 4.2. *If we denote by S the Lie quasi-bialgebroid structure induced by the Poisson quasi-Nijenhuis structure with background $(\pi, N, d\omega_N, -d\omega)$, then $S = e^{-\omega}\circ(e^{-\pi}\mu-\mu)$, or equivalently*

$$S = e^{-\omega}(\mu_\pi),$$

where μ_π is the Lie algebroid structure defined by (3).

In the next proposition we will see that the Poisson quasi-Nijenhuis structure with background $(\pi, N, d\omega_N, -d\omega)$ is induced (as shown in theorem 2.5) by a subtangent structure.

Proposition 4.3. *The $(1,1)$-tensor $J=\begin{pmatrix} N & \pi^\sharp \\ -\omega_N{}^\flat & -{}^tN \end{pmatrix}$ is a subtangent structure (i.e., a c.p.s. structure with $\lambda=0$) on $(A\oplus A^*, \mu-d\omega)$.*

Proof. Using the theorems 4.1 and 2.7 we only need to prove

$$\begin{cases} N^2-\pi^\sharp\circ\omega_N{}^\flat=0, \\ \omega_N{}^\flat\circ N={}^tN\circ\omega_N{}^\flat, \\ 2(i_N\, d\omega_N+\mathcal{H})=d(i_N\,\omega_N). \end{cases}$$

But the verification of the two first conditions is straightforward and, using the fact that $i_N\,\omega_N=i_{N^2}\omega$, the last condition is equivalent to (11). □

In the remaining part of this section, we will see that, in the theorem 4.1, if we impose restrictions on the 2-form ω we get already known structures stronger than Poisson quasi-Nijenhuis with background. We also notice that the pairs $(\pi,\omega),(\pi,N)$ and (ω,N) thus obtained correspond to (or slightly generalize) already known compatible pairs.

Corollary 4.4. [Poisson Nijenhuis] *If $\pi\in\Gamma(\bigwedge^2 A)$ is a Poisson bivector and $\omega\in\Gamma(\bigwedge^2 A^*)$ is a closed 2-form such that $d\omega_N=0$, then (π,N) is a Poisson Nijenhuis structure on A.*

Remarks 4.5. 1. A pair (π,ω) in the conditions of the corollary above is exactly what is called a $P\Omega$-structure in [9].

2. The condition $d\omega_N = 0$ is the compatibility condition for (ω, N) to be a Hitchin pair as it is defined in [2] for $A = TM$. The pair (ω, N) above is more general because ω is not necessarily symplectic.

3. Using the fact that ω is a closed form, we can prove that the compatibility condition $d\omega_N = 0$ is equivalent to two other known compatibility conditions:

   - ω is a complementary 2-form for π as in [17];
   - (ω, N) is a ΩN-structure as in [9].

Let us justify briefly the last remark. In [17], Vaisman defines ω as a complementary 2-form for π when

$$[\omega, \omega]_\pi = 0,$$

where $[.,.]_\pi$ is the natural extension to $\Gamma(\bigwedge^\bullet A^*)$ of the bracket $[.,.]_\pi$ defined in (3). But in terms of big bracket and elements of $C^\infty(T^*\Pi A)$, we have

$$[\omega, \omega]_\pi = \{\{\omega, \{\pi, \mu\}\}, \omega\},$$

and using the Jacobi identity twice we obtain

$$[\omega, \omega]_\pi = 2\{N, \{\mu, \omega\}\} - \{\mu, \{N, \omega\}\},$$

which corresponds to

$$[\omega, \omega]_\pi = 2 i_N d\omega - 2d(\omega_N). \tag{12}$$

In [9], Magri and Morosi define a pair (ω, N) to be a ΩN-structure if a particular 3-form $S(\omega, N)$ vanishes. But we can write

$$S(\omega, N) = -i_N d\omega + d(\omega_N). \tag{13}$$

Therefore, using (12) and (13) the vanishing of $d\omega_N$ is equivalent, when $d\omega = 0$, to the vanishing of $[\omega, \omega]_\pi$ or the vanishing of $S(\omega, N)$.

Corollary 4.6. [Poisson quasi-Nijenhuis] *If $\pi \in \Gamma(\bigwedge^2 A)$ is a Poisson bivector and $\omega \in \Gamma(\bigwedge^2 A^*)$ is a closed 2-form then $(\pi, N, d\omega_N)$ is a Poisson quasi-Nijenhuis structure on A (without background).*

We can also define a Poisson Nijenhuis structure with background (π, N, H) by considering $\psi = 0$ in the definition 2.3. Up to our knowledge, this structure was never studied before. We have the following result.

Corollary 4.7. [Poisson Nijenhuis with background] *If $\pi \in \Gamma(\bigwedge^2 A)$ is a Poisson bivector and $\omega \in \Gamma(\bigwedge^2 A^*)$ is a 2-form such that $d\omega_N = 0$, then $(\pi, N, -d\omega)$ is a Poisson Nijenhuis structure with background on A.*

Observation 4.8. In the above results, the bivector π is a true Poisson bivector. So the last structure we obtain is different from a possible compatibility between a Poisson structure with background [13] and a Nijenhuis tensor.

Acknowledgements. I would like to thank Yvette Kosmann-Schwarzbach for suggesting these topics and for always useful discussions about this work.

References

1. Caseiro, R., De Nicola, A. and Nunes da Costa, J.: On Poisson quasi-Nijenhuis Lie algebroids, math.DG/0806.2467v1.

2. Crainic, M.: Generalized complex structures and Lie brackets, math.DG/0412097v2.

3. Grabowski, J.: Courant-Nijenhuis tensors and generalized geometries, Monografías de la Real Academia de Ciencias de Zaragoza, 29, 101–112, (2006).

4. Gualtieri, M.: Generalized complex geometry, DPhil thesis, Oxford University, (2004), math.DG/0703298.

5. Kosmann-Schwarzbach, Y.: Jacobian quasi-bialgebras and quasi-Poisson Lie groups, Contemp. Math., 132, 459-489, (1992).

6. Kosmann-Schwarzbach, Y.: The Lie bialgebroid of a Poisson-Nijenhuis manifold, Lett. Math. Phys., 38, no. 4, 421–428, (1996).

7. Kosmann-Schwarzbach, Y. :Poisson and symplectic functions in Lie algebroid theory, to appear in Progress in Mathematics, Festschrift in honor of Murray Gerstenhaber and Jim Stasheff.

8. Liu, Z.-J., Weinstein, A. and Xu, P.: Manin triples for Lie bialgebroids, J. Differential Geom., 45, no. 3, 547–574, (1997).

9. Magri, F. and Morosi, C.: A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson-Nijenhuis manifolds, Quaderno S.19, Univ of Milan, (1984).

10. Roytenberg, D.: Courant algebroids, derived brackets and even symplectic supermanifolds, PhD thesis, UC Berkeley, (1999).

11. Roytenberg, D.: Quasi-Lie bialgebroids and twisted Poisson manifolds, Lett. Math. Phys., 61, no. 2, 123-137, (2002).

12. Roytenberg, D.: On the structure of graded symplectic supermanifolds and Courant algebroids, In: T. Voronov (ed.), Quantization, Poisson Brackets and Beyond, Contemp. Math 315, pp. 169–185, Amer. Math. Soc., Providence, (2002).

13. Ševera, P. and Weinstein, A.: Poisson geometry with a 3-form background, Progr. Theor. Phys. Suppl., 144, 145–154, (2001).

14. Stiénon M. and Xu, P.: Poisson quasi-Nijenhuis manifolds, Comm. Math. Phys., 270, no. 3, 709–725, (2007).

15. Terashima, Y.: On Poisson functions, J. Symplectic Geom., 6, no. 1, 1–7, (2008).

16. Vaĭntrob, A. Yu.: Lie algebroids and homological vector fields, Uspekhi Mat. Nauk, 52, no. 2(314), 161–162, (1997).

17. Vaisman, I.: Complementary 2-forms of Poisson structures, Compositio Math., 101, no. 1, 55–75, (1996).

18. Vaisman, I.: Reduction and submanifolds of generalized complex manifolds, Differential Geom. Appl., 25, no. 2, 147–166, (2007).

19. Zucchini, R.: The Hitchin model, Poisson-quasi-Nijenhuis, geometry and symmetry reduction, J. High Energy Phys. , no. 10, 075, 29 pp., (2007).